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Some Results on the Optimal Depletion of Exhaustible Resources Under Negative Discounting

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1. INTRODUCTION

The theory of optimal economic growth, in the form given it by Ramsey (1928), and developed by many others, is primarily concerned with determining the pattern of investment in augmentable capital goods, which is most desirable for an economy, according to a "utilitarian" social welfare function. In these studies, "natural resources" are almost invariably assumed to be supplied exogenously in given amounts, in each period. This approach is clearly unsuitable for examining the optimal pattern of depletion of exhaustible resources.

In recent years, there has been a major concern over the world's dwindling supplies of exhaustible natural resources, and a realization of their increasing significance in production (in the absence of suitable substitutes). This, in turn, has led to a significant literature, concerned with finding the characteristics of optimal growth programs for economies with exhaustible resource constraints; that is, of jointly determining the optimal depletion patterns of such resources, and optimal investment in augmentable capital goods.

It is recognized that technical progress is an economic force which offsets the limitations imposed by exhaustible resources [see, for example, Stiglitz (1974)]. At the same time population growth tends to heighten such limitations [see, for example, Solow (1974), Ingham and Simmons (1975)]. In this paper, we consider a model in which capital, labour and an exhaustible resource produce output, which can be accumulated as capital or consumed. The production function is subject to (exponential) technical progress, and labour (identical to "population" in the model) is growing exponentially.

A planner evaluates per-period social welfare according to a classical Utilitarian index, and following Ramsey (1928), has a zero subjective discount rate. The planner seeks a programme which "maximizes" the sum of per-period social welfares. Since population is growing exponentially, this involves an optimization exercise with a negative "effective" discount rate.

This paper is concerned with finding necessary, and sufficient, conditions for the existence of a "valuation finite" optimal programme in the above framework. [See Section 2 for definitions of concepts used.]

We use three conditions for our existence theorem (Theorem 1). The first conditions states that the effect of technical progress outweighs the effect of population growth (see Condition B.1). The second states that the utility function is bounded (see Condition B.2). The third essentially amounts to a condition on the rate at which utility must approach its upper bound as consumption goes to infinity, in relation to the consumption growth possibilities given by the production function, population growth and technical progress (see Condition B.3).

In Section 4, we establish three necessary conditions for the existence of a valuation finite optimal programme (Theorem 2). First, it is shown that if an optimal programme exists (whether it is valuation finite or not) then the effect of technical progress must outweigh the effect of population growth (in the same sense as Condition B.1). Second, if an optimal programme exists, the utility function must be bounded (Condition B.2). The third condition expresses much the same restriction as (B.3), but in a slightly weaker form (see Condition B.3′).

Examples are given in Sections 3 and 4, which illustrate how Theorems 1 and 2 can be applied to cases where the utility function assumes parametric forms. Remarks following the Theorems relate our necessary, and sufficient conditions for existence of a valuation finite optimal programme, to the results on this issue in the existing literature.

2. THE MODEL

Production

Consider an economy with changing technology, specified by a sequence of production functions, $G_t(t=0,1,2,\ldots)$ from R^3_+ to R_+. A current output level, Z, is producible, at date $(t+1)$, from a capital input, K, exhaustible resource input, D, and labour input, L, available at date t, if $Z=G_t(K,D,L)$.

Capital is assumed to be non-depreciating, and total output, Y, at date $(t+1)$, is then defined as $[G_t(K,D,L)+K]$. A sequence of total-output functions, $F_t(t=0,1,2,\ldots)$ from R^3_+ to R_+ can then be defined, for $(K,D,L)\geqq 0$, by

$$F_t(K,D,L)=G_t(K,D,L)+K. \tag{1}$$

We will consider the case where technical change is exponential, and neutral with respect to the inputs in the sense that there is a scalar $\lambda\geqq 1$, and a function, G, from R^3_+ to R_+, such that

$$G_t(K,D,L)=\lambda^t G(K,D,L) \quad \text{for } t\geqq 0. \tag{2}$$

The function, G, is assumed to satisfy:

Assumption 1. G is continuous, concave, and homogeneous of degree one for $(K,D,L)\geqq 0$; it is differentiable for $(K,D,L)\gg 0$.

Assumption 2. $G_k=(\partial G/\partial K)>0$, $G_D=(\partial G/\partial D)>0$, $G_L=(\partial G/\partial L)>0$ for $(K,D,L)\gg 0$.

The initial capital input $\underset{\sim}{K}$, and the initial available stock of the exhaustible resource, $\underset{\sim}{S}$, are historically given, and positive. The available labour force (identical in this paper to "population") at date t, denoted by $\underset{\sim}{L}_t$, is exogenously given, and grows exponentially. That is, there is a scalar $n>1$, such that

$$\underset{\sim}{L}_t=n^t \quad \text{for } t\geqq 0. \tag{3}$$

A feasible programme is a sequence $\langle K,D,L,Y,C\rangle=\langle K_t,D_t,L_t,Y_{t+1},C_{t+1}\rangle$ satisfying

$$\begin{gathered} K_0=\underset{\sim}{K}, \quad \textstyle\sum_{t=0}^{\infty} D_t\leqq \underset{\sim}{S}, \qquad L_t=\underset{\sim}{L}_t \quad \text{for } t\geqq 0 \\ Y_{t+1}=F_t(K_t,D_t,L_t), \qquad C_{t+1}=Y_{t+1}-K_{t+1} \quad \text{for } t\geqq 0 \\ (K_t,D_t,Y_{t+1},C_{t+1})\geqq 0 \quad \text{for } t\geqq 0. \end{gathered} \tag{4}$$

Given a feasible programme $\langle K, D, L, Y, C\rangle$, we will write

$$\begin{aligned} k_t &= (K_t/L_t), \quad d_t = (D_t/L_t) \quad \text{for } t \geqq 0 \\ c_t &= (C_t/L_t), \quad y_t = (Y_t/L_t) \quad \text{for } t \geqq 1. \end{aligned} \tag{5}$$

Preferences

The planner is endowed with a utility function u from R_+ to R. A feasible programme $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is called optimal if

$$\limsup_{T\to\infty} \sum_{t=1}^{T} [L_t u(c_t) - L_t^* u(c_t^*)] \leqq 0 \tag{6}$$

for every feasible programme $\langle K, D, L, Y, C\rangle$. Since for every feasible programme $\langle K, D, L, Y, C\rangle$, $L_t = \underset{\sim}{L}_t = n^t$ for $t \geqq 0$, so (6) is the same as

$$\limsup_{T\to\infty} \sum_{t=1}^{T} n^t [u(c_t) - u(c_t^*)] \leqq 0. \tag{7}$$

Thus, although the planner's time-preference involves a zero (subjective) discount rate [as is clear from (6)] but his practice of evaluating a period's social welfare by the Classical Utilitarian index $[Lu(c)]$, rather than the Average Utilitarian index $[u(c)]$, together with the fact that the labour force is growing exponentially, amounts to an optimization problem with a discount factor greater than one [as is clear from (7)], or what is the same thing, a discount rate $[(1/n) - 1]$ which is negative.

We will, in some parts of the paper, be concerned with a stronger notion of optimality. We will say that $\tilde{v}$ is a linear transformation of u if there is $(A, b) \in R^2$, $b \neq 0$, such that $\tilde{v}(c) = A + bu(c)$ for $c \geqq 0$. A valuation finite optimal programme $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is an optimal programme such that there is a linear transformation $\tilde{v}$ of u for which

$$\sum_{t=1}^{\infty} L_t \tilde{v}(c_t^*) \quad \text{is convergent.} \tag{8}$$

This requirement is similar to showing that the programme $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is "eligible" [Koopmans (1965)], "good" [Gale (1967)], or "valuation finite" [Hammond and Kennan (1979)].[1]

The following assumptions on u will be used in the paper:

Assumption 3. $u(c)$ is strictly increasing for $c \geqq 0$.

Assumption 4. $u(c)$ is continuous for $c \geqq 0$.

Assumption 5. $u(c)$ is concave for $c \geqq 0$.

Assumption 6. $u(c)$ is differentiable for $c > 0$, and $u'(c) \to \infty$ as $c \to 0$.

3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF A VALUATION FINITE OPTIMAL PROGRAMME

In this section, we will show that under a set of conditions on the utility and production functions, and on the rate of population growth and technical progress, a valuation finite optimal programme exists. For the existence result we must ensure, roughly speaking, that even with exhaustible resource constraints, *per capita* consumption can be increased to infinity fast enough to make the sum of per-period social welfare levels converge. This suggests the need for an interconnecting condition between the production and the utility functions, and we formulate this as Condition B.3.

To get a precise existence result, we will specify the function G to be of the Cobb–Douglas form:

Assumption 7. $G(K, D, L) = K^{\alpha}D^{\beta}L^{\delta}$ for $(K, D, L) \geqq 0$ where $(\alpha, \beta, \delta) \gg 0$ and $\alpha + \beta + \delta = 1$.

The use of Assumption 7 in the exhaustible resource literature is widespread [see, for example, Solow (1974), Stiglitz (1974), Dasgupta and Heal (1974)].

It should be mentioned that under Assumption 7 the model resembles the one studied by Stiglitz (1974, Section 3). However, it should be noted that Stiglitz also specifies the utility function in parametric form, and allows a positive subjective discount rate. In contrast, the procedure followed in this paper is to specify the production function parametrically by Assumption 7, and determine the class of utility functions for which there will exist a valuation finite optimal programme, under a zero subjective discount rate; that is, a negative "effective" discount rate.

Under Assumptions 3–7, we will use three conditions to prove the existence theorem. Firstly, we will need

Condition B.1. $\lambda > n^{\beta}$.

This is a precise way of expressing, in this model, the condition that "the effect of technical progress outweighs the effect of population growth".

Secondly, we will need

Condition B.2. $U \equiv \sup_{c \geqq 0} u(c) < \infty$.

The bounded utility hypothesis may appear to restrict the scope of the existence result. However, it really does not, as I will demonstrate in Section 4 that a *necessary condition* for the existence of an optimal programme is Condition B.2; so is Condition B.1.

Thirdly, we need a condition connecting the utility function to the production possibilities. This can be developed as follows, given Conditions B.1 and B.2. Denote (λ/n^{β}) by m, $m^{[1/(1-\alpha)]}$ by $\bar{g}$, and $[U - u(c)]$ by $v(c)$ for $c \geqq 0$. Let e denote, as usual, the base of the natural logarithms. For $1 < g \leqq \bar{g}$, define a g-effective utility function $w(x; g)$, for $x \geqq e$, by

$$w(x; g) = v[g^{(\log x/\log n')}]. \tag{9}$$

By definition, $w(x; g) > 0$ for $x \geqq e$, and it is a continuous, decreasing function of x. The *area under the g-effective utility function is

$$\int_e^{\infty} w(x; g)\, dx. \tag{10}$$

The third condition we need is

Condition B.3. There is some g, satisfying $1 < g < \bar{g}$, such that the *area under the g-effective utility function is finite.

For our existence theorem, the important result we need is:

Lemma 1. *Under Assumption 7 and Condition B.1, given $1 < g < \bar{g}$, there exists a feasible programme $\langle K, D, L, Y, C\rangle$, and an integer, $1 \leqq T < \infty$, such that*

$$c_t \geqq g^t \quad \textit{for } t \geqq T. \tag{11}$$

Proof. Choose $0 < a < \alpha$, with a sufficiently close to α, such that $G \equiv m^{[1/(1-a)]} > g$. Choose $A > 0$, such that $A^{\beta} = 2Gn$. Choose $\theta > 0$, such that $m^{[\alpha-a)/(1-a)]} = \theta^{\beta}$. Then, since $m>1$ by Condition B.1, so $\theta>1$, and we have

$$\sum_{t=0}^{\infty}[1/\theta^t]<\infty. \tag{12}$$

Choose $\hat{\sigma}>0$, such that

$$\sum_{t=0}^{\infty}[\hat{\sigma}A/\theta^t]=\underset{\sim}{S}. \tag{13}$$

Finally, choose $\sigma=\min[1/2, \hat{\sigma}, \underset{\sim}{K}]$.

Now, define a sequence $\langle K, D, L, Y, C\rangle$ in the following way. Let $K_0=\underset{\sim}{K}$, and $K_t=\sigma G^t n^t$ for $t\geqq 1$; $D_t=[\sigma A/\theta^t]$ for $t\geqq 0$; $L_t=\underset{\sim}{L}_t=n^t$ for $t\geqq 0$; $Y_{t+1}=F_t(K_t, D_t, \tilde{L}_t)$ and $C_{t+1}=Y_{t+1}-K_{t+1}$ for $t\geqq 0$. To show that $\langle K, D, L, Y, C\rangle$ is a feasible programme we have only to show that $C_{t+1}\geqq 0$ for $t\geqq 0$.

For $t\geqq 0$, we have

$$C_{t+1}=F_t(K_t, D_t, L_t)-K_{t+1}=\lambda^t K_t^{\alpha} D_t^{\beta} L_t^{\delta}-K_{t+1}$$

$$\geqq\left[\frac{\lambda^t G^{\alpha t} n^{\alpha t}\sigma^{(\alpha+\beta)}A^{\beta} n^{\delta t}}{\theta^{\beta t}}\right]-\sigma G^{t+1} n^{t+1}$$

$$=\left[\frac{\sigma^{(\alpha+\beta)}A^{\beta} G^t n^t m^{[(\alpha-a)/(1-a)]t}}{\theta^{\beta t}}\right]-\sigma G^{t+1} n^{t+1}$$

[by substituting $G\equiv m^{[1/(1-a)]}$, and simplifying]

$$\geqq 2\sigma G n G^t n^t-\sigma G^{t+1} n^{t+1}$$

{since

$$0<\sigma<1, 0<(\alpha+\beta)<1,$$

and

$$\theta^{\beta}=m^{[(\alpha-a)/(1-a)]}\}=\sigma G^{t+1} n^{t+1}.$$

Hence, $\langle K, D, L, Y, C\rangle$ is a feasible programme, and

$$c_{t+1}\geqq\sigma G^{t+1} \quad \text{for } t\geqq 0. \tag{14}$$

Since $G>g$, so there is an integer $1\leqq T<\infty$, such that (11) holds. ||

Theorem 1. *Under Assumptions* 3, 4, *and* 7, *there exists a valuation finite optimal programme if*

B.1 $\lambda>n^{\beta}$.

B.2 $\sup_{c\geqq 0}u(c)<\infty$

B.3 $\int_e^{\infty}w(x; g)dx<\infty$ *for some g, satisfying* $1<g<\bar{g}$.

Proof. By Lemma 1, there is a feasible programme $\langle K, D, L, Y, C\rangle$, and an integer $1\leqq T<\infty$, such that

$$c_t\geqq g^t \quad \text{for } t\geqq T \tag{15}$$

where g is given by Condition B.3.

We note that $w(x; g)$ is a positive, continuous, decreasing function of x, for $x\geqq e$. Let $\mathscr{T}$ be the smallest integer greater than $[1/\log n]$. Then, for $t\geqq\mathscr{T}$,

$$\int_{n^t}^{n^{t+1}} w(x; g)dx\geqq(n^{t+1}-n^t)w(n^{t+1}; g). \tag{16}$$

Simplifying (16), we have

$$\int_{n^t}^{n^{t+1}} w(x;g)dx \geqq [1-(1/n)]n^{t+1}w(n^{t+1};g). \tag{17}$$

So, using (17) and Condition B.3, we have

$$\sum_{t=\mathcal{T}}^{\infty} n^{t+1}w(n^{t+1};g)<\infty. \tag{18}$$

Using the definition of $w(x;g)$, we have

$$\sum_{t=\mathcal{T}}^{\infty} n^{t+1}v(g^{t+1})<\infty. \tag{19}$$

Since v is a positive, decreasing function, so, using (15), we have

$$\sum_{t=1}^{\infty} n^t v(c_t)<\infty. \tag{20}$$

Then, by Brock and Gale (1969, Lemma 2, p. 236) there is a feasible programme $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ such that

$$\limsup_{T\to\infty} \sum_{t=1}^{T} n^t[v(c_t^*)-v(c_t')] \leqq 0 \tag{21}$$

for every feasible programme $\langle K', D', L', Y', C'\rangle$. Hence, $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is an optimal programme. Using (20) in (21), and $v(c)\geqq 0$ for $c \geqq 0$,

$$\sum_{t=1}^{\infty} n^t v(c_t^*)<\infty. \tag{22}$$

Since $v(c)\geqq 0$ for $c\geqq 0$, so $\langle K^*, D^*, L^*, Y^*, C^*\rangle$ is a valuation finite optimal programme. ||

Remark 1. Note that Theorem 1 does not use the concavity of u.

Remark 2. It is possible to follow the suggestion of Solow (1974, p. 34), and replace Assumption 7 by

Assumption 7*. $G(K, D, L) = H(K, L)D^{\beta}$, where $0<\beta<1$, and H is homogeneous of degree $1-\beta$.

and follow the method used above to get an existence result.[2]

Remark 3. The use of Condition B.3 to obtain (18) is a slight variation of the methods used to obtain the "Maclaurin–Cauchy integral test," and the "Cauchy Condensation test" [see, for example, Hardy (1967, pp. 351–355)].

Example 1. This example demonstrates how Theorem 1 can be applied in a case where the utility function is parametrically specified.

Let $u(c) = -[1/c^{\nu}]$ where $\nu>0$; $\lambda > n^{\beta}$; and $\bar{g}^{\nu}>n$. Then (B.1), (B.2) are clearly satisfied. Also, there is $1<g<\bar{g}$, such that $g^{\nu}>n$ (by choosing g sufficiently close to $\bar{g}$). Then,

$$w(x;g) = [1/g^{(\nu \log x/\log n)}] = [1/x^{(\nu \log g/\log n)}]$$

Since $g^{\nu}>n$, so $[\nu \log g/\log n]>1$, and

$$\int_e^{\infty} w(x;g)\,dx<\infty$$

Hence Condition B.3 is satisfied. So, by Theorem 1, there exists a valuation finite optimal programme.

Example 2. Theorem 1 also enables us to construct a utility function, such that there will exist a valuation finite optimal programme, irrespective of the actual values of $\bar{g}$ and n, provided Condition B.1 holds. We give an example of such a utility function.

Let $u(c) = -[1/e^c]$; $\lambda > n^\beta$. Then Conditions B.1 and B.2 are satisfied. Pick any g satisfying $1 < g < \bar{g}$. Then,

$$w(x; g) = [1/\exp\{g^{(\log x/\log n)}\}] = [1/\exp\{x^{(\log g/\log n)}\}].$$

Since $\log g > 0$, and $\log n > 0$, so $N \equiv [\log g/\log n] > 0$. Hence, we have

$$\int_e^\infty w(x; g)dx = \int_e^\infty [1/e^{x^N}]dx < \infty.$$

Hence, Condition B.3 is satisfied, and by Theorem 1, there exists a valuation finite optimal programme.

4. NECESSARY CONDITIONS FOR THE EXISTENCE OF A VALUATION FINITE OPTIMAL PROGRAMME

It is worthwhile to show that the existence theorem of Section 3 was not obtained under overly strong sufficient conditions. This can best be demonstrated by proving that if a valuation finite optimal programme exists, then Conditions B.1, B.2 and B.3 are satisfied. Although, we have not been able to do precisely this, it is possible to demonstrate that if an optimal programme exists then Conditions B.1 and B.2 are satisfied. Also, if a valuation finite optimal programme exists, then a condition "close to" Condition B.3 is also satisfied. We write this condition as follows:

Condition B.3′. The *area under the $\bar{g}$-effective utility function is finite.

Lemma 2. *Under Assumptions* 3–7, *if there exists an optimal programme then Condition B.*1 *holds.*

Proof. If there is an optimal programme $\langle K, D, L, Y, C\rangle$, then for each $t \geqq 1$, the expression

$$n^t u\left[\frac{F_{t-1}(K_{t-1}, D_{t-1}, L_{t-1}) - K}{n^t}\right] + n^{t+1} u\left[\frac{F_t(K, D_t, L_t) - K_{t+1}}{n^{t+1}}\right]$$

must be a maximum at $K = K_t$. By Assumption 6, $C_t > 0$ for $t \geqq 1$, and so by Assumption 7, $(K_t, D_t) \gg 0$ for $t \geqq 0$. So, using Assumptions 6 and 7,

$$u'(c_t) = u'(c_{t+1})(\partial F_t/\partial K_t) \quad \text{for } t \geqq 1. \tag{23}$$

$(\partial F_t/\partial K_t) > 1$ for $t \geqq 1$, so by (23) and Assumption 5,

$$c_t < c_{t+1} \quad \text{for } t \geqq 1. \tag{24}$$

Since $c_t > 0$ for $t \geqq 1$, so by (24), $\mu \equiv \inf_{t \geqq 1} c_t > 0$.

By feasibility, we know that for $t \geqq 0$,

$$k_{t+1} = (m^t/n)k_t^\alpha D_t^\beta + (1/n)k_t - c_{t+1}. \tag{25}$$

If Condition B.1 does not hold, then $m \leqq 1$, so

$$k_{t+1} \leqq (1/n)k_t^\alpha D_t^\beta + (1/n)k_t - c_{t+1}. \tag{26}$$

Since $D_t \leqq \underset{\sim}{S}$, so (26) yields

$$k_{t+1} \leqq (\underset{\sim}{S}^\beta/n)k_t^\alpha + (1/n)k_t - c_{t+1} \tag{27}$$

Since $n>1$, and $0<\alpha<1$, so (27) implies that there is $V<\infty$, such that $k_t \leqq V$ for $t \geqq 0$. Also, $D_t \to 0$ as $t \to \infty$, so there is $T<\infty$, such that for $t \geqq T$,

$$(1/n)k_t^\alpha D_t^\beta \leqq (V^\alpha/n)D_t^\beta \leqq (\mu/2).$$

Hence, for $t \geqq T$, using (26),

$$k_{t+1} \leqq (\mu/2)+(1/n)k_t-\mu=(1/n)k_t-(\mu/2) \tag{28}$$

Hence, $k_{t+1} \leqq (1/n)k_t$ for $t \geqq T$, and so $k_t \to 0$ as $t \to \infty$. Using this in (28), $k_t<0$ for large t, a contradiction. Hence, Condition B.1 must hold. ||

Lemma 3. *Under Assumptions* 3–7, *if an optimal programme exists, then Condition* B.2 *holds.*

Proof. If there is an optimal programme $\langle K, D, L, Y, C\rangle$, then for each $t \geqq 1$, and $0 \leqq D \leqq D_{t-1}+D_t$, the expression

$$n^t u\left[\frac{F_{t-1}(K_{t-1}, D, L_{t-1})-K_t}{n^t}\right]+n^{t+1}u\left[\frac{F_t(K_t, D_{t-1}+D_t-D, L_t)-K_{t+1}}{n^{t+1}}\right]$$

must be a maximum at $D=D_{t-1}$. By Assumption 6, $C_t>0$ for $t \geqq 1$, and so by Assumption 7, $(K_t, D_t) \gg 0$ for $t \geqq 0$. So, using Assumptions 6 and 7

$$u'(c_t)(\partial F_{t-1}/\partial D_{t-1})=u'(c_{t+1})(\partial F_t/\partial D_t) \quad \text{for } t \geqq 1. \tag{29}$$

By Lemma 2, c_t is monotonically increasing by (24). We claim that $c_t \to \infty$ as $t \to \infty$. If not, then c_t is bounded above, and $u'(c_t)$ is bounded below by a positive number. So, by (29), $(\partial F_t/\partial D_t)$ is bounded above. We have

$$(\partial F_t/\partial D_t)=[(\beta\lambda^t K_t^\alpha D_t^\beta L_t^\delta)/D_t] \quad \text{for } t \geqq 0. \tag{30}$$

Since $(\partial F_t/\partial D_t)$ is bounded above, and $D_t \to 0$ as $t \to \infty$, so $\lambda^t G(K_t, D_t, L_t) \to 0$ as $t \to \infty$. By feasibility, we have for $t \geqq 0$,

$$K_{t+1}=\lambda^t G(K_t, D_t, L_t)+K_t-C_{t+1}. \tag{31}$$

By (24), $C_{t+1} \geqq \mu n^{t+1} \geqq \mu$ for $t \geqq 0$. Since $\lambda^t G(K_t, D_t, L_t) \to 0$ as $t \to \infty$, so there is $T<\infty$, such that for $t \geqq T$, $\lambda^t G(K_t, D_t, L_t) \leqq (\mu/2)$. Hence, by (31), for $t \geqq T$,

$$K_{t+1} \leqq K_t-(\mu/2). \tag{32}$$

But then $K_t<0$ for large t, a contradiction. This establishes our claim that $c_t \to \infty$ as $t \to \infty$.

Define a sequence $\langle p_t, q_t, \gamma_t\rangle$ as follows:

$$\begin{aligned} &p_0=u'(c_1)(\partial F_0/\partial K_0); \qquad p_t=u'(c_t) \quad \text{for } t \geqq 1 \\ &q_t=u'(c_1)(\partial F_0/\partial D_0) \quad \text{for } t \geqq 0 \\ &\gamma_t=p_{t+1}(\partial F_t/\partial L_t) \quad \text{for } t \geqq 0. \end{aligned} \tag{33}$$

It is easy to check that $W(C, L) \equiv Lu(c)$ is concave in C, and $(\partial W/\partial C)=u'(c)$. Thus, for $C \geqq 0$, we have

$$W(C, L_t)-W(C_t, L_t) \leqq u'(c_t)(C-C_t). \tag{34}$$

Rearranging terms in (34) yields

$$W(C_t, L_t)-p_tC_t \geqq W(C, L_t)-p_tC \quad \text{for } C \geqq 0, t \geqq 1. \tag{35}$$

Also, using the concavity of F_t in (K, D, L),

$$F_t(K, D, L) - F_t(K_t, D_t, L_t) \leqq (\partial F_t/\partial K_t)(K - K_t) + (\partial F_t/\partial D_t)(D - D_t) + (\partial F_t/\partial L_t)(L - L_t). \tag{36}$$

Multiplying through by p_{t+1}, and using (23), (29), (33),

$$p_{t+1}[F_t(K, D, L) - F_t(K_t, D_t, L_t)] \leqq p_t(K - K_t) + q_t(D - D_t) + \gamma_t(L - L_t). \tag{37}$$

Rearranging terms in (37) and using Assumption 7,

$$\begin{aligned} 0 &= p_{t+1}F_t(K_t, D_t, L_t) - p_tK_t - q_tD_t - \gamma_tL_t \\ &\geqq p_{t+1}F_t(K, D, L) - p_tK - q_tD - \gamma_tL \quad \text{for } (K, D, L) \geqq 0, \quad t \geqq 0. \end{aligned} \tag{38}$$

We will use (35) and (38) to show that p_tK_t is bounded above. For this purpose, we write, for $t \geqq 0$,

$$\begin{aligned} p_{t+1}C_{t+1} = p_{t+1}Y_{t+1} - p_{t+1}K_{t+1} &= p_{t+1}Y_{t+1} - p_tK_t - q_tD_t - \gamma_tL_t \\ &\quad + q_tD_t + \gamma_tL_t + [p_tK_t - p_{t+1}K_{t+1}] \\ &= q_tD_t + \gamma_tL_t + [p_tK_t - p_{t+1}K_{t+1}], \end{aligned}$$

by using (38). Now,

$$\gamma_tL_t = p_{t+1}(\partial F_t/\partial L_t)L_t = [(\partial F_t/\partial L_t)L_t/\partial F_t/\partial D_t]q_t$$

[by using (29)] $= D_t[(\partial F_t/\partial L_t)L_t/(\partial F_t/\partial D_t)D_t]q_t = (\delta/\beta)q_tD_t$. Hence,

$$\textstyle\sum_{t=0}^{T} p_{t+1}C_{t+1} = p_0K_0 + \sum_{t=0}^{T} q_tD_t + \sum_{t=0}^{T}(\delta/\beta)q_tD_t - p_{T+1}K_{T+1}. \tag{39}$$

So,

$$\begin{aligned} \textstyle\sum_{t=0}^{T} p_{t+1}C_{t+1} &\leqq p_0K_0 + q_0[1 + (\delta/\beta)] \textstyle\sum_{t=0}^{T} D_t \\ &\leqq p_0K_0 + q_0[1 + (\delta/\beta)]\underset{\sim}{S}. \end{aligned}$$

Hence, $\sum_{t=1}^{\infty} p_tC_t < \infty$. Using this in (39) shows that p_tK_t is convergent, and bounded above by $p_0K_0 + q_0[1 + (\delta/\beta)]\underset{\sim}{S}$.

Now, we will show that $u(c)$ is bounded above. For $t \geqq 1$,

$$L_t[u(C_{t+1}/L_t) - u(c_t)] = W(C_{t+1}, L_t) - W(C_t, L_t) \leqq p_t(C_{t+1} - C_t). \tag{48}$$

Also, since $L_{t+1} > L_t \geqq 1$, and c_t is increasing with t,

$$u(c_{t+1}) - u(c_t) \leqq L_t[u(c_{t+1}) - u(c_t)] \leqq L_t[u(C_{t+1}/L_t) - u(c_t)]. \tag{41}$$

Combining (40) and (41),

$$u(c_{t+1}) - u(c_t) \leqq p_t(C_{t+1} - C_t). \tag{42}$$

Now,

$$\begin{aligned} p_t(C_{t+1} - C_t) &= p_t(Y_{t+1} - K_{t+1}) - p_t(Y_t - K_t) \\ &= p_tY_{t+1} - p_{t-1}K_t - q_{t-1}D_t - \gamma_{t-1}L_t + q_{t-1}D_t + \gamma_{t-1}L_t + p_{t-1}K_t - p_tK_{t+1} \\ &\quad - [p_tY_t - p_{t-1}K_{t-1} - q_{t-1}D_{t-1} - \gamma_{t-1}L_{t-1} \\ &\quad + q_{t-1}D_{t-1} + \gamma_{t-1}L_{t-1} + p_{t-1}K_{t-1} - p_tK_t] \\ &\leqq q_{t-1}(D_t - D_{t-1}) + \gamma_{t-1}(L_t - L_{t-1}) + [p_{t-1}K_t - p_tK_{t+1}] - [p_{t-1}K_{t-1} - p_tK_t] \end{aligned}$$

by using (38). Now, $\gamma_{t-1}(L_t - L_{t-1}) \leqq \gamma_{t-1}L_t \leqq n\gamma_{t-1}L_{t-1}$, and $q_{t-1}(D_t - D_{t-1}) \leqq q_{t-1}D_t = q_tD_t$. So we have

$$p_t(C_{t+1} - C_t) \leqq q_tD_t + n\gamma_{t-1}L_{t-1} + [p_{t-1}K_t - p_tK_{t+1}] - [p_{t-1}K_{t-1} - p_tK_t]. \tag{43}$$

Thus, summing (43) from $t = 1$ to $t = T$

$$\textstyle\sum_{t=1}^T p_t(C_{t+1} - C_t) \leqq \sum_{t=1}^T q_tD_t + n\sum_{t=1}^T \gamma_{t-1}L_{t-1} + [p_0K_1 - p_TK_{T+1}] - [p_0K_0 - p_TK_T].$$

We have already shown that $\gamma_tL_t = (\delta/\beta)q_tD_t$. So,

$$\textstyle\sum_{t=1}^T p_t(C_{t+1} - C_t) \leqq q_0[1 + n(\delta/\beta)]\underset{\sim}{S} + p_0K_1 + p_TK_T. \tag{44}$$

Using (42) and (44),

$$u(c_{T+1}) - u(c_1) \leqq q_0[1 + n(\delta/\beta)]\underset{\sim}{S} + p_0K_1 + p_TK_T. \tag{45}$$

Since p_TK_T is bounded above, so is $u(c_{T+1})$, by (45). But we have shown that $c_t \to \infty$ as $t \to \infty$, so $u(c)$ is bounded above, and Condition B.2 holds. ||

Theorem 2. *Under Assumptions* 3–7, *if there exists a valuation finite optimal programme, then*

(B.1) $\lambda > n^\beta$

(B.2) $\sup_{c \geqq 0} u(c) < \infty$

(B.3′) $\int_e^\infty w(x; \bar{g})dx < \infty.$

Proof. If there exists a valuation finite optimal programme $\langle K, D, L, Y, C\rangle$, then it is also an optimal programme. So, by Lemmas 2 and 3, Conditions B.1 and B.2 must hold.

Consider the sequence $\langle X_t\rangle$ given by $X_0 = \underset{\sim}{K}$, $X_{t+1} = \lambda^t X_t^\alpha \underset{\sim}{S}^\beta L_t^\delta + X_t$. Denote (X_t/L_t) by x_t. Note that $X_{t+1} > X_t$, and so $nx_{t+1} > x_t$ for $t \geqq 0$. Then, for $t \geqq 0$

$$nx_{t+1} = m^t x_t^\alpha \underset{\sim}{S}^\beta + x_t. \tag{46}$$

Using (46), and $nx_{t+1} > x_t$, we have

$$x_{t+1}^{1-\alpha} - x_t^{1-\alpha} \leqq [nx_{t+1}/x_t^\alpha] - x_t^{1-\alpha} = m^t\underset{\sim}{S}^\beta. \tag{47}$$

Summing from $t = 0$ to $t = T$ in (47)

$$x_{T+1}^{1-\alpha} - x_0^{1-\alpha} \leqq \underset{\sim}{S}^\beta \textstyle\sum_{t=0}^T m^t \leqq [\underset{\sim}{S}^\beta/(m-1)]m^{T+1}. \tag{48}$$

Consequently, there is $\hat{E} < \infty$, such that

$$x_t^{1-\alpha} \leqq \hat{E}m^t \quad \text{for } t \geqq 0. \tag{49}$$

And, there is $E < \infty$, such that

$$x_t \leqq Em^{[t/(1-\alpha)]} = E\bar{g}^t. \tag{50}$$

It is clear from the definition of the sequence $\langle X_t\rangle$, that $C_t \leqq X_t$ for $t \geqq 1$, so

$$c_t \leqq E\bar{g}^t. \tag{51}$$

Now, since $\langle K, D, L, Y, C\rangle$ is a valuation finite optimal programme, so there is a linear transformation $\tilde{v}$ of u, such that

$$\textstyle\sum_{t=1}^\infty L_t\tilde{v}(c_t) \quad \text{is convergent.} \tag{52}$$

Since c_t monotonically increases to infinity (by Lemma 3), so $u(c_t)$ converges to U as $t \to \infty$. Then, writing

$$\tilde{v}(c) = A + bu(c) = b[(A/b) + u(c)],$$

we note that by (52), $(A/b) = -U$. Hence, (52) implies that

$$\textstyle\sum_{t=1}^{\infty} L_t v(c_t) < \infty \tag{53}$$

where $v(c) = U - u(c)$ for $c \geqq 0$.

Choose $\hat{A} > 0$ such that $\bar{g}^{(\log \hat{A}/\log n)} = E$. Let τ be the smallest integer such that $\hat{A}n^{\tau} \geqq e$. Then, since $w(x; \bar{g})$ is a positive, continuous decreasing function of x, so for $t \geqq \tau$,

$$\begin{aligned}\int_{\hat{A}n^t}^{\hat{A}n^{t+1}} w(x; \bar{g})dx &\leqq \hat{A}(n^{t+1} - n^t)w(\hat{A}n^t; \bar{g}) \\ &= \hat{A}(n-1)n^t w(\hat{A}n^t; \bar{g}) \\ &= \hat{A}(n-1)n^t v(E\bar{g}^t) \\ &\leqq \hat{A}(n-1)n^t v(c_t), \quad \text{by (51).}\end{aligned}$$

Hence, by using (53), Condition B.3′ must hold. ||

Remark 4. It is interesting to note that even if the effect of technical progress just offsets the effect or population growth ($\lambda = n^{\beta}$) there does not exist an optimal programme. In particular, if there is no technical progress ($\lambda = 1$), no optimal programme exists. This is the result obtained by Ingham and Simmons (1975).

Remark 5. It should be noted that Lemma 3 demonstrates that with commonly used utility functions like $u(c) = \sqrt{c}$, or $u(c) = \log c$, no optimal programme can exist in this framework.

Example 3. This example demonstrates how Theorem 2 can be applied in a case where the utility function is parametrically specified.

Let $u(c) = -[1/c^{\nu}]$ where $\nu > 0$; $\lambda > n^{\beta}$; and $\bar{g}^{\nu} \leqq n$. Then Conditions B.1, B.2 are satisfied. Also, we have

$$w(x; \bar{g}) = [1/\bar{g}^{(\nu \log x/\log n)}] = [1/x^{(\nu \log \bar{g}/\log n)}].$$

Since $\bar{g}^{\nu} \leqq n$, so $[\nu \log \bar{g}/\log n] \leqq 1$, and

$$\lim_{x\to\infty} \int_e^x w(x; \bar{g})dx = \infty.$$

Hence, Condition B.3′ is violated. Hence, by Theorem 2 there does not exist a valuation finite optimal programme.[3]

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NOTES

1. Actually, valuation finiteness in the sense of Hammond and Kennan would involve $\sum_{t=1}^{\infty} (\bar{u}_t - u(c_t^*))L_t$ converging, where $\bar{u}_t$ is the supremum over all feasible programmes of $u(c_t)$. Note that if the utility function is not bounded above, $\bar{u}_t$ will still exist, because there is a bound on capital and resources at time $t-1$ (given $\underset{\sim}{K}$ and $\underset{\sim}{S}$), and so on output and consumption at time t. And even if the utility function is bounded above (which,

incidentally, I show later to be a necessary condition for the existence of an optimal programme) by $\bar{u}$, one could have valuation finiteness in the sense of Hammond and Kennan, without having valuation finiteness in my sense (since $\bar{u}_t < \bar{u}$ for $t \geqq 1$). Thus, in both respects, their definition is somewhat more general than mine.

2. Note that with Assumption 7, and technical progress involving geometric progression, the notion of Hicks-neutral technical progress (which I assume) coincides with the notions of capital augmenting (Solow neutral), labour augmenting (Harrod neutral) or even resource augmenting progress. With the more general form given by Assumption 7*, one can introduce capital and labour augmenting progress in $H(K, L)$, following Brock and Gale (1969), and resource augmenting progress "separately". In contrast to Assumption 7, this generalization would illustrate the importance of the rates of technical progress associated with different factors (compared to n) in a condition analogous to Condition B.1.

3. It is fairly easy to show that in this example, $\bar{g}^{\nu} \leqq n$ implies that there is no optimal programme. For if there did exist an optimal programme, then following the proof of Lemma 3 we have $\sum_{t=1}^{\infty} p_t C_t < \infty$, or using (4.11), $\sum_{t=1}^{\infty} L_t u'(c_t) c_t < \infty$. Using the special form of the utility function, we then have

$$\sum_{t=1}^{\infty} L_t(-u(c_t)) < \infty, \quad \text{or} \quad \sum_{t=1}^{\infty} L_t v(c_t) < \infty,$$

which means there is a valuation finite optimal programme, a contradiction to what we have established in Example 4.1. Thus, Examples 1 and 3 show that for $u(c) = -(1/c^{\nu})$, the conditions $\lambda > n^{\beta}$ and $\bar{g}^{\nu} > n$ are necessary and sufficient for the existence of an optimal programme. Also, these conditions are necessary and sufficient for the existence of valuation finite optimal programmes.

Examples 1 and 3 can be used to compare our existence results with those obtained by Brock and Gale (1969). From (11) and (50) the "asymptotic growth factor" in this model, in the sense of Brock and Gale, is $\bar{g}$. The asymptotic elasticity of u is $(-\nu)$. Brock and Gale's criterion for existence of an optimal programme would then be $n\bar{g}^{(-\nu)} < 1$, which agrees with our result in Example 1. Their criterion for non-existence of an optimal programme would be $n\bar{g}^{(-\nu)} > 1$. Example 3 shows that this is too strong a criterion for non-existence, and demonstrates that our criteria are capable of handling "borderline" cases like $n\bar{g}^{(-\nu)} = 1$, where Brock and Gale's criteria would fail. This is because the criteria of Brock and Gale involve only the asymptotic values of key magnitudes; our criteria involve, in addition, the rate at which these asymptotic values are approached.

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